Problem Set 5

This problem set – the last one purely on discrete mathematics – is designed as a cumulative review of the topics we've covered so far and a proving ground to try out your newfound skills with mathematical induction. The problems here span all sorts of topics – parallel processing, the nature of infinity, tiling problems, and social networks – and we hope that it serves as a fitting coda to our whirlwind tour of discrete math!

We recommend that you ***read Handout #28, "Guide to Induction," before starting this problem set***. It contains a lot of useful advice about how to approach problems inductively, how to structure inductive proofs, and how to not fall into common inductive traps. Additionally, before submitting, be sure to ***read over Handout #29, the "Induction Proofwriting Checklist,"*** for a list of specific things to watch for in your solutions before submitting.

As a note on this problem set – normally, you're welcome to use any proof technique you'd like to prove results in this course. On this problem set, we've specifically requested on some problems that you prove a result inductively. For those problems, you should prove those results using induction or complete induction, even if there is another way to prove the result. (If you'd like to use induction in conjunction with other techniques like proof by contradiction or proof by contrapositive, that's perfectly fine.)

As always, please feel free to drop by office hours, visit Piazza, or send us emails if you have any questions. We'd be happy to help out.

Good luck, and have fun!

Due Friday, February 16th at 2:30PM.
There is no checkpoint problem.

Problem One: Chains and Antichains

Let A be an arbitrary set and $<_A$ be some strict order over A. A ***chain in*** $<_A$ is a series $x_1, \ldots, x_k$ of elements drawn from A such that

$$x_1 <_A x_2 <_A \ldots <_A x_k.$$

Intuitively, a chain is a series of values in ascending order according to the strict order $<_A$. The ***length*** of a chain is the number of elements in that chain.

i. Consider the $\subsetneq$ relation over the set $\wp(\{a, b, c\})$, where $A \subsetneq B$ means that $A \subseteq B$ but $A \neq B$. What is the length of the longest chain in this strict order? Give an example of a chain with that length. No justification is necessary.

Draw the Hasse diagram and see if you can find a visual intuition for the definition of a chain.

Now, let's cover a new definition. An ***antichain in*** $<_A$ is a set $X \subseteq A$ such any two elements in X are incomparable by the $<_A$ relation. In other words, a set $X \subseteq A$ is an antichain if

$$\forall a \in X. \forall b \in X. (a \nless_A b \quad \land \quad b \nless_A a)$$

The ***size*** of an antichain X is the number of elements in X.

ii. Consider the $\subsetneq$ relation over the set $\wp(\{a, b, c\})$. What is the size of the largest antichain in this strict order? Give an example of an antichain with that size. No justification is necessary.

Draw the Hasse diagram and see if you can find a visual intuition for the definition of an antichain.

Given an arbitrary strictly ordered set, you can't say anything *a priori* about the size of the largest chain or antichain in that strict order. However, you can say that at least one of them must be relatively large relative to the strictly ordered set.

Let $<_A$ be an arbitrary strict order over an arbitrary set A containing exactly n^2+1 elements for some natural number $n \geq 1$. We're going to ask you to prove the following result: either A contains a chain of length $n+1$ or an antichain of size $n+1$ (or both). Following the advice from Handout 17, we'll prove this by instead proving that if A does *not* contain a chain of length $n+1$ or greater, then A must contain an antichain of size $n+1$ or greater.

iii. For each element $a \in A$, we'll say that the ***height*** of a is the length of the longest chain whose final element is a. Prove that if A does not contain a chain of length $n+1$ or greater, then there must be at least $n+1$ elements of A at the same height.

Something to think about: what's the smallest possible height of an element of A?

iv. Your result from part (iii) establishes that if A does not contain a chain of length $n+1$ or greater, there must be a collection of $n+1$ elements of A at the same height as one another. Prove that if A does not contain a chain of length $n+1$ or greater, then it contains an antichain of size $n+1$ or greater.

Intuitively speaking, if $<_A$ is a strict order over A that represents some prerequisite structure on a group of tasks, a chain represents a series of tasks that have to be performed one after the other, and an antichain represents a group of tasks that can all be performed in parallel (do you see why?) In the context of parallel computing, the result you've proved states that if a group of tasks doesn't contain long dependency chains, that group must have a good degree of parallelism. Take CS149 for more information!

Problem Two: Recurrence Relations

A *recurrence relation* is a recursive definition of the terms in a sequence. Typically, a recurrence relation specifies the value of the first few terms in a sequence, then defines the remaining terms from the previous terms. For example, the *Fibonacci sequence* can be defined by the following recurrence relation:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_{n+2} = F_n + F_{n+1}$$

The first terms of this sequence are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, etc.

Some recurrence relations define well-known sequences. For example, consider the following recurrence relation:

$$a_0 = 1$$
$$a_{n+1} = 2a_n$$

The first few terms of this sequence are 1, 2, 4, 8, 16, 32, …, which happen to be powers of two. It turns out that this isn't a coincidence – this recurrence relation perfectly describes the powers of two.

i. Prove by induction that for any $n \in \mathbb{N}$, we have $a_n = 2^n$.

In case you're wondering what you're asked to prove here: the official definition a_n is given by the recurrence relation. If you ever need to determine what the value of a_n is for some value of n, look back at that definition. We want you to prove that, as a consequence of that definition, the value of a_n, the nth term in the series, is always exactly 2^n.

Minor changes to the recursive step in a recurrence relation can lead to enormous changes in what numbers are generated. Consider the following two recurrence relations, which are similar to the a_n sequence defined above but with slight changes to the recursive step:

$$b_0 = 1 \qquad\qquad c_0 = 1$$
$$b_{n+1} = 2b_n - 1 \qquad\qquad c_{n+1} = 2c_n + 1$$

ii. Find non-recursive definitions for b_n and c_n, then prove by induction that your definitions are correct.

This one is hard to do just by eyeballing the recurrences. Try expanding out the first few terms of these sequences and see what you find.

Finding non-recursive definitions for recurrences (often called "solving" the recurrence) is useful in the design and analysis of algorithms. Commonly, when trying to analyze the runtime of an algorithm, you will arrive at a recurrence relation describing the runtime on an input of size *n* in terms of the runtime on inputs of smaller sizes. Solving the recurrence then lets you precisely determine the runtime. To learn more, take CS161, Math 108, or consider reading through the excellent textbook *Concrete Mathematics* by Graham, Knuth, and Patashnik.

Problem Three: It'll All Even Out

Our very first proof by induction was the proof that for any natural number n, we have that

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1.$$

This result is still true for the case where $n = 0$, since in that case the sum on the left-hand side of the equation is the *empty sum* of zero numbers, which is by definition equal to zero. It's also true for the case where $n = 1$; in that case, the sum on the left-hand side of the equality just has a single term in it (2^0) and the right-hand side has the same value.

Below is a proof by complete induction of an incorrect statement about what happens when you sum up zero or more real numbers:

Theorem: The sum of any number of real numbers is even.

Proof: Let $P(n)$ be the statement "the sum of any n real numbers is even." We will prove by complete induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.

As a base case, we prove $P(0)$, that the sum of any 0 real numbers is even. The sum of any zero numbers is the empty sum and is by definition equal to 0, which is even. Thus $P(0)$ holds.

For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(0)$, …, and $P(k)$ are true. We will prove that $P(k+1)$ is true, meaning that the sum of any $k+1$ real numbers is even. To do so, let x_1, x_2, …, x_k, and x_{k+1} be arbitrary real numbers and consider the sum

$$x_1 + x_2 + \ldots + x_k + x_{k+1}.$$

We can group the first k terms and the last term independently to see that

$$x_1 + x_2 + \ldots + x_k + x_{k+1} = (x_1 + x_2 + \ldots + x_k) + (x_{k+1}).$$

Now, consider the sum $x_1 + x_2 + \ldots + x_k$ of the first k terms. This is the sum of k real numbers, so by our inductive hypothesis that $P(k)$ is true we know that this sum must be even. Similarly, consider the sum x_{k+1} consisting of just the single term x_{k+1}. By our inductive hypothesis that $P(1)$ is true, we know that this sum must be even.

Overall, we have shown that $x_1 + x_2 + \ldots + x_k + x_{k+1}$ can be written as the sum of two even numbers (namely, $x_1 + x_2 + \ldots + x_k$ and x_{k+1}), so $x_1 + x_2 + \ldots + x_k + x_{k+1}$ is even. Thus $P(k+1)$ is true, completing the induction. ■

Of course, this result has to be incorrect, since there are many sums of real numbers that don't evaluate to an even number. The sum $2 + 3 + 4$, for example, works out to 9, and the sum $\pi + 1$ doesn't even work out to an integer!

What's wrong with this proof? Be as specific as possible. For full credit, you should be able to identify a specific claim made in the proof that is not correct, along with an explanation as to why it's incorrect.

Think about our "induction as a machine" analogy from lecture that explains why you can start with a base case and inductive step and end up with a proof that works for all natural numbers. See what happens if you try that out here.

Problem Four: So What Exactly Is Multiplication, Anyway?

On Problem Set Four, you proved that $2\aleph_0 = \aleph_0$ by finding a bijection between $\mathbb{N} \times \{★, ☺\}$ and $\mathbb{N}$. But why exactly did finding that bijection tell you anything about $2\aleph_0$?

Let's suppose you have two cardinal numbers κ_1 and κ_2. (A ***cardinal number*** is a quantity that represents the size of a set; all natural numbers are cardinal numbers, as is $\aleph_0$.) We can define $\kappa_1 \cdot \kappa_2$ as follows: pick any sets A and B where $|A| = \kappa_1$ and $|B| = \kappa_2$, then determine $|A \times B|$. The resulting cardinal number is then defined to be $\kappa_1 \cdot \kappa_2$. In other words, $4 \cdot 3$ is *defined* to be $|A \times B|$ for any sets A and B where $|A| = 4$ and $|B| = 3$. Similarly, $2\aleph_0$ is *defined* to be $|A \times B|$ for any sets A and B where $|A| = 2$ and $|B| = \aleph_0$.

For this definition to work, we have to make sure that the cardinality of the Cartesian product depends purely on the cardinalities of the two sets, not their contents. For example, this definition wouldn't give us a way to compute $4 \cdot 3$ if the cardinality of the Cartesian product of a set of four apples and three oranges was different than the cardinality of the Cartesian product of a set of four figs and three dates. We need to show that for any sets A, B, C, and D, that if $|A| = |C|$ and $|B| = |D|$, then $|A \times B| = |C \times D|$. That way, when determining $\kappa_1 \cdot \kappa_2$, it doesn't matter which sets of cardinality κ_1 and κ_2 we pick. (Contrast thiswith $|A \cup B|$: if you have sets A and B, you can't necessarily predict $|A \cup B|$ from $|A|$ and $|B|$.)

Let A, B, C, and D be arbitrary sets where $|A| = |C|$ and $|B| = |D|$. Our goal is to prove $|A \times B| = |C \times D|$. Since we know $|A| = |C|$, there has to be some bijection $g : A \to C$. Since we know $|B| = |D|$, there has to be some bijection $h : B \to D$. Now, consider the function $f : A \times B \to C \times D$ defined as follows:

$$f(a, b) = (g(a), h(b))$$

That is, the output of f when applied to the pair (a, b) is an ordered pair whose first element is $g(a)$ and whose second element is $h(b)$.

i. Using the function f defined above, prove that $|A \times B| = |C \times D|$. Specifically, prove that f is a bijection between $A \times B$ and $C \times D$.

Two ordered pairs are equal if and only if their corresponding elements are equal. Although the preceding discussion talked about how to multiply cardinal numbers, that discussion only works because of the result that you'll be proving here, so you can't assume that $|A \times B| = |A| \cdot |B|$ in the course of writing this proof. Your proof should purely focus on proving that f is a bijection.

We can define the ***Cartesian power*** of a set as follows. For any set A and any positive natural number n, we define A^n inductively:

$$A^1 = A$$
$$A^{n+1} = A \times A^n \text{ (for } n \geq 1\text{)}$$

Amazingly, we know that $|\mathbb{N}| = |\mathbb{N}^2|$, meaning that there's the same number of pairs of natural numbers as there are natural numbers themselves. Feel free to use this fact in the following problem.

ii. Using your result from (i), the above definition, and the fact that $|\mathbb{N}| = |\mathbb{N}^2|$, prove by induction that $|\mathbb{N}^k| = |\mathbb{N}|$ for all nonzero $k \in \mathbb{N}$. This result means that for any nonzero finite k, there are the same number of k-tuples of natural numbers as natural numbers.

If κ is a cardinal number and $n \geq 1$ is a natural number, then *by definition* the value of κ^n is $|A^n|$, where A is any set of cardinality κ. Your result from part (ii) shows that $\aleph_0{}^n = \aleph_0$ for any positive natural number n.

Problem Five: Induction and Recursion

There's a close connection between mathematical induction and recursion, and many of the proofs by induction that we did in class can be thought of as claims about how specific recursive functions work.

One of the first proofs by induction that we did was to prove that, given a collection of 3^n coins containing a single counterfeit coin that's heavier than the rest, it is always possible to discover which coin is fake using exactly n weighings on a balance. The key idea behind the proof was, essentially, a recursive algorithm that can be used to actually go and find which of the coins is counterfeit!

i. Implement a recursive function

```
Coin counterfeitIn(std::vector<Coin> coins, Balance balance);
```

that takes as input a set of ***exactly*** $\boldsymbol{3^n}$ ***coins*** for some natural number n, one of which is counterfeit and weighs more than the rest, and returns which one that is. You're provided a balance you can use to weigh groups of coins and can make at most n weighings on that balance. Check the header `CounterfeitCoins.h` for a description of the relevant types here.

Test your code locally on your machine before submitting it, since if your solution crashes due to a logic error the autograder won't give you any useful feedback. Our provided starter files provide an interface you can use to test out your function on a number of different inputs and will show you which coins actually get weighed against one another.

Your code from part (i) shows that the *inductive* argument we made in class can be converted into a *recursive* function that actually finds the coin!

Now, here's a fun little variant on the counterfeit coin problem. Imagine that you're given a collection of coins. You're told that there *might* be a counterfeit in it, but then again, there might not be. If there is a counterfeit coin, it's guaranteed to be heavier than the rest. Your job is to determine whether there even is a counterfeit coin at all and, if so, to return which one it is.

ii. Implement a recursive function

```
Coin maybeCounterfeitIn(std::vector<Coin> coins, Balance balance)
```

that takes as input a set of ***exactly*** $\boldsymbol{3^n - 1}$ ***coins*** for some natural number n, which *might* contain a counterfeit that weighs more than the rest. The function should either return the counterfeit coin if one exists, or return the special constant `None` if none of the coins are counterfeit. You're provided a balance you can use to weigh groups of coins and can make at most n weighings on that balance.

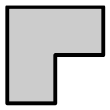
Again, test locally, and test thoroughly – it's easy to miss cases!

iii. Using the recursive intuition that you developed in the course of solving part (ii) of this problem, prove that given any collection of exactly $3^n - 1$ coins, of which at most one is a counterfeit that weighs more than the rest, it is always possible to identify which coin that is using at most n weighings on a balance (or to report that all coins are genuine). Your proof should have a similar structure to the one about counterfeit coins from lecture. While you should should not explicitly reference the code you wrote in part (ii) of this problem, you may want to use the same recursive insight from that problem to guide the structure of your proof.

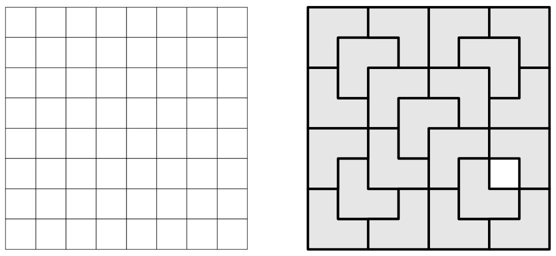
We hope that this exercise gives you a better sense for the interplay between theory (proof by induction) and practice (recursive problem-solving). If you're interested in this sort of thing, we strongly recommend checking out CS161, where you'll alternate between designing clever algorithms and using induction to prove that they work correctly.

Problem Six: Tiling with Triominoes

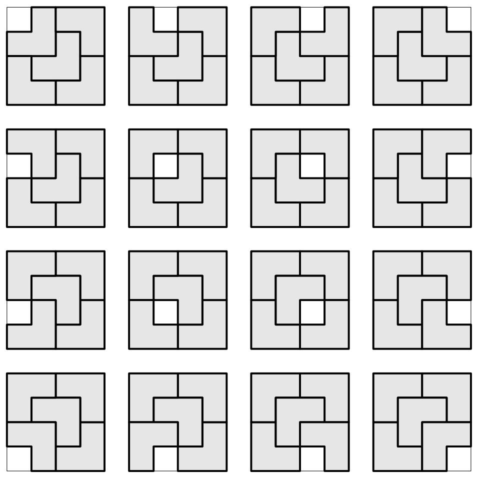
Recall from Problem Set One that a *right triomino* is an L-shaped tile that looks like this:



Suppose you're given a $2^n \times 2^n$ grid of squares and want to tile it with right triominoes by covering the grid with triominoes such that all triominoes are completely on the grid and no triominoes overlap. Here's an attempt to cover an 8×8 grid with triominoes, which doesn't manage to cover all squares:



Amazingly, it turns out that it is always possible to tile any $2^n \times 2^n$ grid that's missing exactly one square with right triominoes. It doesn't matter what n is or which square is removed; there is always a solution to the problem. For example, here are all the ways to tile a 4×4 grid that has a square missing:



This question explores why this is the case.

i. Prove, by induction, that $4^n - 1$ is a multiple of three for any $n \in \mathbb{N}$.

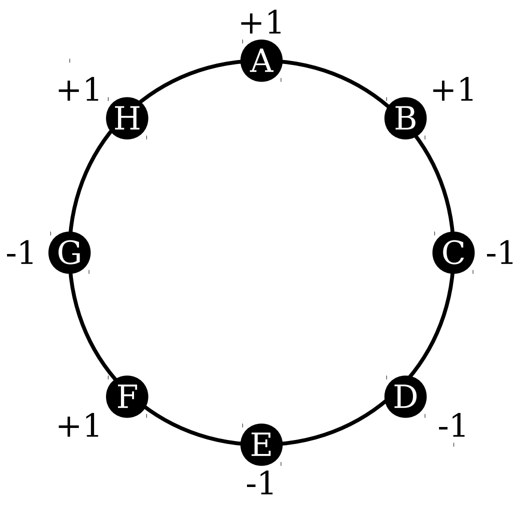
Any $2^n \times 2^n$ grid missing a square has a number of squares has exactly $4^n - 1$ squares, and so its number of squares is a multiple of three. Although you *can* show that a figure *can't* be tiled with triominoes by showing that its number of squares isn't a multiple of three, you *can't* show that a figure *can* be tiled with triominoes purely by showing that its number of squares is a multiple of three. The arrangement matters.

ii. Draw a figure made of squares where the number of squares is a multiple of three, yet the figure cannot be tiled with right triominoes. Briefly justify your answer; no formal proof is necessary.

iii. Prove by induction that for any natural number n, any $2^n \times 2^n$ grid with any one square removed can be tiled by right triominoes.

Before you write this proof, try seeing if you can find a nice recursive pattern you can follow that will let you fully tile any such board. Once you've found it, formalize your idea in your answer. You may want to think about how to start with a larger board and subdivide it into some number of smaller boards.

Problem Seven: The Circle Game



Here's a game you can play. Suppose that you have a circle with $2n$ arbitrarily-chosen points on its circumference. n of these points are labeled +1, and the remaining n are labeled -1. One sample circle with eight points, of which four are labeled +1 and four are labeled -1, is shown to the right.

Here's the rules of the game. First, choose one of the $2n$ points as your starting point. Then, start moving clockwise around the circle. As you go, you'll pass through some number of +1 points and some number of -1 points. You lose the game if at any point on your journey you pass through more -1 points than +1 points. You win the game if you get all the way back around to your starting point without losing.

For example, if you started at point A, the game would go like this:

Start at A: +1.
Pass through B: +2.
Pass through C: +1.
Pass through D: 0.
Pass through E: -1. *(You lose.)*

If you started at point G, the game would go like this:

Start at G: -1 *(You lose.)*

However, if you started at point F, the game would go like this:

Start at F: +1.
Pass through G: 0.
Pass through H: +1.
Pass through A: +2.
Pass through B: +3.
Pass through C: +2.
Pass through D: +1.
Pass through E: +0.
Return to F. *(You win!)*

Interestingly, it turns out that no matter which n points are labeled +1 and which n points are labeled -1, there is always at least one point you can start at to win the game.

Prove, by induction, that the above fact is true for any $n \geq 1$.

This one is all about finding the right setup. Check the Guide to Induction and Inductive Proofwriting Checklist for details.

Problem Eight: Nim

Nim is a family of games played by two players. The game begins with several piles of stones, each of which has zero or more stones in it, that are shared between the two players. Players alternate taking turns removing any nonzero number of stones from any single pile of their choice. If at the start of a player's turn all the piles are empty, then that player loses the game.

Prove, by induction, that if the game is played with just two piles of stones, each of which begins with exactly the same number of stones, then the second player can always win the game if she plays correctly.

Before trying to write up your answer to this question, we recommend playing this game with a partner until you can find a winning strategy. Once you spot the pattern, see if you can find a way to formalize it using induction. Be wary of writing statements of the form "and so on" or "by repeating this;" those aren't rigorous ways to formalize that a process will eventually do something.

Problem Nine: Independent Sets Revisited

Recall from Problem Set Four that an ***independent set*** in a graph $G = (V, E)$ is a set $I \subseteq V$ where no two nodes in I are adjacent to one another. As you saw on that problem set, many problems in graph theory boil down to finding large independent sets in a graph. The question then arises: how big of an independent set can you reasonably expect to find?

i. Let G be a graph where each node has degree less than or equal to d. (As a reminder, the ***degree*** of a node in a graph is the number of nodes that it's adjacent to.) Prove, by ***complete*** induction on n, that if G is a graph with $n \geq 0$ nodes, then G has an independent set of size at least $\frac{n}{d+1}$.

As with the Circle Game problem, one of the big steps in solving this one is making sure you're setting up this problem properly. Will you start with a smaller graph and add another node in, or start with a larger graph and take a note out?

We strongly recommend checking your work by thinking about how you'd convert your inductive proof into a recursive algorithm, then testing that algorithm out on some sample graphs. If you've found the right argument, you should end up with a rather simple algorithm.

There's also no guarantee that the graph G has a node of degree exactly d. The statement "everyone in CS103 has height less than or equal to 1km" is true, even though no one is anywhere close to that tall.

Graphs are often used to model social networks: each person is a node, and friendships are represented by edges. The sorts of graphs you find this way in the real world have all sorts of interesting properties (take CS224W or CS267 for more details!) In particular, social networks tend to have a lot of ***triangles***, collections of three nodes that are all mutually adjacent.

This question explores a class of graphs that are quite different from the graphs that typically arise in social networks: ***triangle-free graphs***. A triangle-free graph is one that contains no triangles. That is, if you pick three distinct nodes in the graph, some two of them will not be adjacent.

ii. Draw a triangle-free graph with nine nodes that has a node of degree eight. Justify your answer.

iii. Prove that if G is a triangle-free graph with n^2 nodes, then G contains an independent set of size at least n.

This last one is tricky, so don't worry if you don't see it immediately. Note that we didn't say to use induction here. You may want to find a way to work in the result from part (i) of this problem. Work backwards: what value of d would you need to have in order to apply that result? Don't forget the result you found in part (ii), though: triangle-free graphs can have very high-degree nodes. What happens then?

Optional Fun Problem: Synchronicity (1 Point Extra Credit)

Let's say that an ***era*** is a historical time period with a definitive start date and definitive end date. For example, the Meiji Era ran from October 23, 1868 to July 30, 1912, and the Cuban Missile Crisis ran from October 16, 1962 to October 28, 1962. For simplicity, we'll assume that these time ranges include the entirety of their start and end dates.

Prove that no matter how you choose any fifty eras from history, you can either (1) find a date that's contained in at least eight of those eras, or (2) find eight eras of which no two have any days in common.